

Equivariant Metaplectic-c Prequantization of Symplectic Manifolds with Hamiltonian Torus Actions

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Abstract

This paper determines a condition that is necessary and sufficient for a metaplectic-c prequantizable symplectic manifold with an effective Hamiltonian torus action to admit an equivariant metaplectic-c prequantization. The condition is evaluated at a fixed point of the momentum map, and is shifted from the one that is known for equivariant prequantization line bundles.

Given a metaplectic-c prequantized symplectic manifold with a Hamiltonian energy function, the author previously proposed a condition under which a regular value of the function should be considered a quantized energy level of the system. This definition naturally generalizes to regular values of the momentum map for a Hamiltonian torus action. We state the generalized definition for such a system, and use an equivariant metaplectic-c prequantization to determine its quantized energy levels.

1 Introduction

Metaplectic-c quantization was introduced by Hess [4] and further developed by Robinson and Rawnsley [6]. It is a generalization of the Kostant-Souriau quantization procedure with half-form correction that applies to a strictly broader class of symplectic manifolds. The starting point for this paper is a symplectic manifold (M, ω) that admits a metaplectic-c prequantization.

Suppose there is a Hamiltonian G action on (M, ω) for some Lie group G . Broadly speaking, a prequantization bundle for (M, ω) is called an equivariant prequantization if the G action lifts to the prequantization bundle in a manner that preserves all of its structures. This concept has been applied to prequantization line bundles in the context of a torus action and an arbitrary compact Lie group action [2]. It has also been applied to spin-c structures in the context of a circle action [1] and a torus action [3].

In this paper, we assume that (M, ω) is metaplectic-c prequantizable and has an effective Hamiltonian torus action with momentum map $\Phi : M \rightarrow \mathfrak{t}^*$, where \mathfrak{t}^* is the dual of the Lie algebra \mathfrak{t} for the torus. Section 2 contains our conventions for Hamiltonian torus actions, and reviews the definitions of the metaplectic-c group and a metaplectic-c prequantization.

In Section 3, we further assume that the torus action has at least one fixed point z . We give the definition of an equivariant metaplectic-c prequantization, and we determine a condition on the value of the momentum map at z that is necessary and sufficient for (M, ω) to admit an equivariant metaplectic-c prequantization. For an equivariant prequantization line bundle, a comparable result

is known [2]: the value $\frac{1}{h}\Phi(z)$ must be in the integer lattice of \mathfrak{t}^* , where $h = 2\pi\hbar$ is Planck's constant. The condition that we obtain is shifted from this due to the lift of the torus action to the symplectic frame bundle for (M, ω) . The statement of our equivariance condition is in Theorem 3.1.

In an earlier paper [7], we defined a quantized energy level for the metaplectic-c prequantized system (M, ω, H) , where $H \in C^\infty(M)$ is viewed as a Hamiltonian energy function on (M, ω) . This definition has a natural generalization to families of Poisson-commuting functions. In particular, in Section 4, we apply it to the components of the momentum map Φ for the torus action. Given an equivariant metaplectic-c prequantization for the system (M, ω, Φ) , we show that the regular values x of Φ that are quantized are exactly those such that $\frac{1}{h}x$ lies in the integer lattice. This is Theorem 4.2. The section concludes by demonstrating that if the torus acts freely on the level set corresponding to a quantized energy level, then the symplectic reduction is metaplectic-c prequantizable.

Lastly, in Section 5, we consider two examples. First, we obtain the quantized energy levels for a harmonic oscillator of arbitrary dimension. The result includes the half-shift predicted by the standard quantum mechanical calculation. Then we consider the complex projective space \mathbb{CP}^2 with an action of the two-dimensional torus T^2 that is induced from a linear T^2 action on \mathbb{C}^3 . We determine the shift in the momentum map required to satisfy the equivariance condition, and find the quantized energy levels. This example is notable because \mathbb{CP}^2 admits a metaplectic-c prequantization but not a metaplectic structure, meaning that quantization results for this system cannot be duplicated using Kostant-Souriau quantization with the half-form correction.

2 Hamiltonian Torus Actions and Metaplectic-c Prequantization

Section 2.1 sets up our notation and conventions for the torus and the momentum map. In Section 2.2, we summarize the basic elements of metaplectic-c prequantization. Considerably more detail, including proofs, were given by Robinson and Rawnsley [6]. A similar review also appears in [7].

2.1 Hamiltonian torus actions

Let T^k be a k -dimensional torus with Lie algebra \mathfrak{t} . Write $\tau \in T^k$ as (τ_1, \dots, τ_k) where each $\tau_j \in U(1)$. Let $\{\xi_1, \dots, \xi_k\}$ be the standard basis for \mathbb{R}^k , and identify \mathfrak{t} with \mathbb{R}^k such that for any $\xi = \sum_{j=1}^k a_j \xi_j \in \mathbb{R}^k$,

$$\exp(\xi) = (e^{2\pi i a_1}, \dots, e^{2\pi i a_k}).$$

Let (M^{2n}, ω) be a connected symplectic manifold, where $n \geq k$. For the remainder of the paper, we assume that T^k has an effective Hamiltonian action on M with momentum map $\Phi : M \rightarrow \mathfrak{t}^*$. For all $\xi \in \mathfrak{t}$, we define the vector field ξ_M on M by

$$\xi_M(m) = \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) \cdot m, \quad \forall m \in M.$$

Our convention for the momentum map is

$$d\Phi^\xi = \xi_M \lrcorner \omega, \quad \forall \xi \in \mathfrak{t}.$$

For each $\xi \in \mathfrak{t}$, denote the flow of ξ_M on M by ϕ_ξ^t . Explicitly,

$$\phi_\xi^t(m) = \exp(t\xi) \cdot m, \quad \forall m \in M,$$

and ϕ_ξ^t is a symplectomorphism for all t . For any $\xi \in \mathfrak{t}$, the action of the element $\tau = \exp(\xi) \in T^k$ on M is given by the map ϕ_ξ^1 . In particular, if $\xi \in \mathbb{Z}^k \subset \mathfrak{t}$, then ϕ_ξ^1 is the identity map on M .

2.2 Metaplectic-c prequantization

Fix a model $2n$ -dimensional symplectic vector space (V, Ω) , together with a compatible complex structure J on V . The symplectic group for (V, Ω) is denoted by $\mathrm{Sp}(V)$. The metaplectic group is the connected double cover $\mathrm{Mp}(V) \xrightarrow{\sigma} \mathrm{Sp}(V)$. The metaplectic-c group is defined to be

$$\mathrm{Mp}^c(V) = \mathrm{Mp}(V) \times_{\mathbb{Z}_2} U(1).$$

We will make use of the following two group homomorphisms on $\mathrm{Mp}^c(V)$. The **projection map** σ appears in the short exact sequence

$$1 \rightarrow U(1) \rightarrow \mathrm{Mp}^c(V) \xrightarrow{\sigma} \mathrm{Sp}(V) \rightarrow 1$$

and restricts to the double covering map on $\mathrm{Mp}(V)$. The **determinant map** η appears in the short exact sequence

$$1 \rightarrow \mathrm{Mp}(V) \rightarrow \mathrm{Mp}^c(V) \xrightarrow{\eta} U(1) \rightarrow 1$$

and acts on $\lambda \in U(1) \subset \mathrm{Mp}^c(V)$ by $\eta(\lambda) = \lambda^2$. The Lie algebra $\mathfrak{mp}^c(V)$ can be identified with $\mathfrak{sp}(V) \oplus \mathfrak{u}(1)$ under $\sigma_* \oplus \frac{1}{2}\eta_*$.

For any $g \in \mathrm{Sp}(V)$, let

$$C_g = \frac{1}{2}(g - JgJ).$$

Then C_g commutes with J , so it is a complex linear map on V . It can be shown [6] that $\mathrm{Det}_{\mathbb{C}} C_g \neq 0$ for all $g \in \mathrm{Sp}(V)$.

We define an embedding of $\mathrm{Mp}^c(V)$ into $\mathrm{Sp}(V) \times \mathbb{C}$ such that each $a \in \mathrm{Mp}^c(V)$ is mapped to the pair $(g, \mu) \in \mathrm{Sp}(V) \times \mathbb{C}$, where $\sigma(a) = g$ and $\eta(a) = \mu^2 \mathrm{Det}_{\mathbb{C}} C_g$. To resolve the ambiguity in the sign of μ , we assume that $I \in \mathrm{Mp}^c(V)$ is mapped to $(I, 1)$. Following [6], we refer to (g, μ) as the **parameters** of $a \in \mathrm{Mp}^c(V)$. Note that if $a \in \mathrm{Mp}(V) = \ker \eta$, then the parameters (g, μ) of a satisfy $\mu^2 \mathrm{Det}_{\mathbb{C}} C_g = 1$.

The unitary group $U(V) \subset \mathrm{Sp}(V)$ is the maximal compact subgroup of $\mathrm{Sp}(V)$, and consists of exactly those elements of $\mathrm{Sp}(V)$ that commute with the complex structure J . For any $g \in U(V)$, $C_g = g$ and $\mathrm{Det}_{\mathbb{C}} C_g = \mathrm{Det}_{\mathbb{C}} g \in U(1)$.

We view the **symplectic frame bundle** $\mathrm{Sp}(M, \omega) \xrightarrow{\rho} (M, \omega)$ as a right principal $\mathrm{Sp}(V)$ bundle over M , defined fiberwise such that for all $m \in M$, every $b \in \mathrm{Sp}(M, \omega)_m$ is a linear symplectic isomorphism $b : (V, \Omega) \rightarrow (T_m M, \omega_m)$. The group $\mathrm{Sp}(V)$ acts on the fibers of $\mathrm{Sp}(M, \omega)$ by precomposition.

Definition 2.1. Let (M, ω) be a symplectic manifold with symplectic frame bundle $\mathrm{Sp}(M, \omega) \xrightarrow{\rho} (M, \omega)$. A **metaplectic-c prequantization** for (M, ω) is a triple (P, Σ, γ) , where:

- $P \xrightarrow{\Pi} M$ is a right principal $\mathrm{Mp}^c(V)$ bundle;
- the map $P \xrightarrow{\Sigma} \mathrm{Sp}(M, \omega)$ satisfies $\rho \circ \Sigma = \Pi$ and $\Sigma(q \cdot a) = \Sigma(q) \cdot \sigma(a)$ for all $q \in P$ and $a \in \mathrm{Mp}^c(V)$;
- γ is a $\mathfrak{u}(1)$ -valued one-form on P , invariant under the principal $\mathrm{Mp}^c(V)$ action, such that $d\gamma = \frac{1}{i\hbar} \Pi^* \omega$, and for all $\alpha \in \mathfrak{mp}^c(V)$, if α generates the vector field α_P on P , then $\gamma(\alpha_P) = \frac{1}{2} \eta_* \alpha$.

If $(P, \Sigma, \gamma) \rightarrow (M, \omega)$ is a metaplectic-c prequantization, then $(P, \gamma) \xrightarrow{\Sigma} \mathrm{Sp}(M, \omega)$ is a principal circle bundle with connection one-form. The circle that acts on the fibers of P is the center $U(1) \subset \mathrm{Mp}^c(V)$.

3 Equivariant Metaplectic-c Prequantization

3.1 Initial constructions

From now on, we assume that (M, ω) is metaplectic-c prequantizable, and we fix a metaplectic-c prequantization (P, Σ, γ) . We have the bundle projection maps $P \xrightarrow{\Pi} M$ and $\text{Sp}(M, \omega) \xrightarrow{\rho} M$.

Recall from Section 2.1 that there is a Hamiltonian T^k action on (M, ω) with momentum map Φ . Each $\xi \in \mathfrak{t}$ generates the vector field ξ_M on M with flow ϕ_ξ^t . Since ϕ_ξ^t preserves ω , it can be lifted to a flow $\tilde{\phi}_\xi^t$ on $\text{Sp}(M, \omega)$, defined by

$$\tilde{\phi}_\xi^t(b) = \phi_{\xi*}^t|_m \circ b, \quad \forall m \in M, \quad \forall b \in \text{Sp}(M, \omega)_m.$$

The corresponding vector field on $\text{Sp}(M, \omega)$ is

$$\tilde{\xi}_M(b) = \left. \frac{d}{dt} \right|_{t=0} \tilde{\phi}_\xi^t(b).$$

Let $\tau = \exp(\xi) \in T^k$ act on $\text{Sp}(M, \omega)$ by $\tilde{\phi}_\xi^1$. It is easily verified that this definition yields a well-defined group action of T^k on $\text{Sp}(M, \omega)$ that lifts the T^k action on M and commutes with the principal $\text{Sp}(V)$ action.

Suppose there is a lift of the T^k action on $\text{Sp}(M, \omega)$ to one on P that preserves γ . Then this T^k action also commutes with the principal $\text{Mp}^c(V)$ action. Let $\xi \in \mathfrak{t}$ generate the vector field ξ_P on P . It is immediate that $L_{\xi_P} \gamma = 0$ if and only if $d(\gamma(\xi_P)) = -\frac{1}{i\hbar} d\Pi^* \Phi^\xi$. If, in particular,

$$\gamma(\xi_P) = -\frac{1}{i\hbar} \Pi^* \Phi^\xi,$$

then (P, Σ, γ) is called an **equivariant metaplectic-c prequantization** for (M, ω, Φ) . An equivalent definition for spin-c structures appears in [1], although it is stated in terms of equivariant cohomology classes. An analogous definition for an equivariant prequantization line bundle appears in [2].

In the remainder of this section, we determine a necessary and sufficient condition for the metaplectic-c prequantized system $(P, \Sigma, \gamma) \rightarrow (M, \omega, \Phi)$ to admit an equivariant metaplectic-c prequantization. For all $\xi \in \mathfrak{t}$, let ξ_P be the vector field on P such that ξ_P is a lift of ξ_M and $\gamma(\xi_P) = -\frac{1}{i\hbar} \Pi^* \Phi^\xi$. Let ψ_ξ^t be the flow of ξ_P on P . If there is a T^k action on P such that $\xi \in \mathfrak{t}$ generates the vector field ξ_P , then $\tau = \exp(\xi) \in T^k$ must act on P by the map ψ_ξ^1 . We will find a condition that ensures that these maps ψ_ξ^1 yield a well-defined T^k action on P . It suffices to guarantee that for all $\xi \in \mathbb{Z}^k \subset \mathfrak{t}$, ψ_ξ^1 is the identity map on P .

The following argument is based on Example 6.10 in [2] (pp. 93-94), which establishes a necessary and sufficient condition for (M, ω, Φ) to admit an equivariant prequantization line bundle. Our application of their proof to a metaplectic-c prequantization requires some additional steps concerning the symplectic frame bundle.

Assume that the T^k action has a fixed point $z \in M$. For example, it is sufficient to assume that M is compact. However, noncompact examples also exist, and the remainder of the argument does not require compactness. In Section 3.2, we will determine a condition on $\Phi(z)$ such that there is a well-defined T^k action on the fiber P_z . Then, in Section 3.3, we will show that this condition guarantees a T^k action on all of P .

3.2 T^k action over a fixed point

Let $z \in M$ be a fixed point of the T^k action. Any $\tau = \exp(\xi) \in T^k$ acts on TM by the pushforward $\phi_{\xi*}^1$, which preserves the symplectic form. In particular, $\tau : T_z M \rightarrow T_z M$ is a linear symplectic isomorphism.

Let U be a neighborhood of z in M over which $P|_U$ admits a local trivialization: $P|_U \cong U \times \text{Mp}^c(V)$. This induces a local trivialization $\text{Sp}(M, \omega)|_U \cong U \times \text{Sp}(V)$, where the map Σ is identified with (Id_U, σ) .

$$\begin{array}{ccc} P|_U & \xrightarrow{\cong} & U \times \text{Mp}^c(V) \\ \downarrow \Sigma & & \downarrow (\text{Id}_U, \sigma) \\ \text{Sp}(M, \omega)|_U & \xrightarrow{\cong} & U \times \text{Sp}(V) \end{array}$$

It further induces a local trivialization $TM|_U \cong U \times V$, under which there is an identification of the symplectic vector space $(T_z M, \omega_z)$ with (V, Ω) . Since T^k acts on $(T_z M, \omega_z)$ by linear symplectic isomorphisms, this identification yields a group homomorphism $T^k \xrightarrow{\kappa_z} \text{Sp}(V)$. By a suitable adjustment of the choice of trivializing section of $P|_U$, we can arrange that

$$T^k \xrightarrow{\kappa_z} U(V) \subset \text{Sp}(V).$$

For emphasis: this property is only required to hold over the single point z . On the level of tangent spaces, we obtain identifications

$$T_z M \cong V, \quad T_{(z, I)} \text{Sp}(M, \omega) \cong V \times \mathfrak{sp}(V), \quad T_{(z, I)} P \cong V \times \mathfrak{mp}^c(V) = V \times \mathfrak{sp}(V) \oplus \mathfrak{u}(1).$$

Let $\xi \in \mathfrak{t}$ be arbitrary. It is immediate that $\xi_M(z) = 0$. The pushforward $\phi_{\xi*}^t$, acting on $T_z M$, becomes the symplectic group element $\kappa_z(\exp(t\xi)) \in \text{Sp}(V)$. The lifted flow $\tilde{\phi}_\xi^t$ on $\text{Sp}(M, \omega)$ satisfies

$$\tilde{\phi}_\xi^t(z, I) = (z, \kappa_z(\exp(t\xi))),$$

and so

$$\tilde{\xi}_M(z, I) = \left. \frac{d}{dt} \right|_{t=0} \tilde{\phi}_\xi^t(z, I) = (0, \kappa_{z*} \xi).$$

The vector field ξ_P is the lift of $\tilde{\xi}_M$ to P such that $\gamma(\xi_P) = -\frac{1}{i\hbar} \Pi^* \Phi^\xi$. At (z, I) , we have

$$\xi_P(z, I) = \left(0, \kappa_{z*} \xi \oplus -\frac{1}{i\hbar} \Phi^\xi(z) \right).$$

The flow ψ_ξ^t of ξ_P satisfies

$$\psi_\xi^t(z, I) = \left(z, \exp \left[t \left(\kappa_{z*} \xi \oplus -\frac{1}{i\hbar} \Phi^\xi(z) \right) \right] \right).$$

The desired action of T^k on P_z exists if and only if $\psi_\xi^1(z, I) = (z, I)$ for all $\xi \in \mathbb{Z}^k \subset \mathfrak{t}$.

Since $U(1)$ is the center of $\text{Mp}^c(V)$, we can write the $\text{Mp}^c(V)$ term in the above expression for $\psi_\xi^t(z, I)$ as

$$\exp \left[t \left(\kappa_{z*} \xi \oplus -\frac{1}{i\hbar} \Phi^\xi(z) \right) \right] = \exp(t\kappa_{z*} \xi \oplus 0) e^{2\pi i t \Phi^\xi(z)/h},$$

where $h = 2\pi\hbar$, and where $\exp(t\kappa_{z*} \xi \oplus 0) \in \text{Mp}(V) \subset \text{Mp}^c(V)$ and $e^{2\pi i t \Phi^\xi(z)/h} \in U(1) \subset \text{Mp}^c(V)$.

The parameters of $\exp(t\kappa_{z*}\xi \oplus 0) \in \text{Mp}(V)$ take the form $(\kappa_z(\exp(t\xi)), \mu(t))$ where

$$\mu(t)^2 \text{Det}_{\mathbb{C}} C_{\kappa_z(\exp(t\xi))} = 1.$$

Note that $t = 0$ corresponds to $I \in \text{Mp}(V)$, so we must have $\mu(0) = 1$. Further, since $\kappa_z(\exp(t\xi)) \in U(V)$, we have $C_{\kappa_z(\exp(t\xi))} = \kappa_z(\exp(t\xi))$ and $\text{Det}_{\mathbb{C}} \kappa_z(\exp(t\xi)) \in U(1)$. Let Δ denote the map from $\text{Sp}(V)$ to \mathbb{C} given by $g \mapsto \text{Det}_{\mathbb{C}} C_g$. Let Δ also denote the restriction of this map to $U(V)$, where it is given by $g \mapsto \text{Det}_{\mathbb{C}} g$, and note that $U(V) \xrightarrow{\Delta} U(1)$ is a group homomorphism.

$$\begin{array}{ccccc} T^k & \xrightarrow{\kappa_z} & U(V) & \xrightarrow{\subset} & \text{Sp}(V) \\ & \searrow w_z & \downarrow \Delta = \text{Det}_{\mathbb{C}} & & \downarrow \Delta = \text{Det}_{\mathbb{C}} C_g \\ & & U(1) & \xrightarrow{\subset} & \mathbb{C} \end{array}$$

Let $w_z = \Delta \circ \kappa_z$. Then

$$\mu(t)^2 w_z(\exp(t\xi)) = 1$$

and $\mu(0) = 0$, which implies that $\mu(t) = w_z(\exp(t\xi))^{-1/2}$. Thus the parameters of $\exp(t\kappa_{z*}\xi \oplus 0)$ are

$$(\kappa_z(\exp(t\xi)), w_z(\exp(t\xi))^{-1/2}).$$

Identify $\mathfrak{u}(1)$ with \mathbb{R} such that for all $\lambda \in \mathfrak{u}(1)$, $\exp(\lambda) = e^{2\pi i \lambda} \in U(1)$. Then $w_z(\exp(t\xi)) = e^{2\pi i t w_{z*}\xi} \in U(1)$, and so $w_z(\exp(t\xi))^{-1/2} = e^{-\pi i t w_{z*}\xi}$. Thus the parameters of $\exp(t\kappa_{z*}\xi \oplus 0)$ are

$$(\kappa_z(\exp(t\xi)), e^{-\pi i t w_{z*}\xi}),$$

which implies that the parameters of $\exp[t(\kappa_{z*}\xi \oplus -\frac{1}{i\hbar}\Phi^\xi(z))]$ are

$$\left(\kappa_z(\exp(t\xi)), e^{-\pi i t w_{z*}\xi} e^{2\pi i t \Phi^\xi(z)/\hbar}\right).$$

Now assume that $\xi \in \mathbb{Z}^k \subset \mathfrak{t}$, and set $t = 1$. The condition $\psi_\xi^1(z, I) = (z, I)$ is satisfied if and only if

$$\left(\kappa_z(\exp(\xi)), e^{-\pi i w_{z*}\xi} e^{2\pi i \Phi^\xi(z)/\hbar}\right) = (I, 1).$$

It is clear that $\kappa_z(\exp(\xi)) = I$, and it remains to ensure that $e^{-\pi i w_{z*}\xi} e^{2\pi i \Phi^\xi(z)/\hbar} = 1$. This equation holds if and only if

$$-\pi i w_{z*}\xi + \frac{2\pi i \Phi^\xi(z)}{\hbar} = 2\pi i N$$

for some $N \in \mathbb{Z}$, which rearranges to

$$\frac{1}{\hbar} \Phi^\xi(z) - \frac{1}{2} w_{z*}\xi = N.$$

Since an equation of this form must hold for all $\xi \in \mathbb{Z}^k \subset \mathfrak{t}$, we conclude that the value $\Phi(z) \in \mathfrak{t}^*$ must satisfy

$$\frac{1}{\hbar} \Phi(z) - \frac{1}{2} w_{z*} \in \mathbb{Z}^{k*} \subset \mathfrak{t}^*.$$

This is similar to, but shifted from, the result in [2] that an equivariant prequantization line bundle exists if and only if $\frac{1}{\hbar} \Phi(z)$ is in the integer lattice $\mathbb{Z}^{k*} \subset \mathfrak{t}^*$ (adjusted for differing \hbar conventions).

3.3 T^k action on P

We continue to follow a modified version of the argument in Example 6.10 of [2]. Fix $\xi \in \mathbb{Z}^k \subset \mathfrak{t}$. Then ψ_ξ^1 is a lift of the identity maps $\tilde{\phi}_\xi^1$ on $\text{Sp}(M, \omega)$ and ϕ_ξ^1 on M .

- Since ψ_ξ^1 is a lift of the identity map on M , there is a map $R_\xi : M \rightarrow \text{Mp}^c(V)$ such that $\psi_\xi^1(q) = q \cdot R_\xi(\Pi(q)) \ \forall q \in P$.
- Since ψ_ξ^1 is a lift of the identity map on $\text{Sp}(M, \omega)$, there is a map $\tilde{R}_\xi : \text{Sp}(M, \omega) \rightarrow U(1) \subset \text{Mp}^c(V)$ such $\psi_\xi^1(q) = q \cdot \tilde{R}_\xi(\Sigma(q)) \ \forall q \in P$.

These observations together imply that the target of the map R_ξ is $U(1)$. That is, there is a map $R_\xi : M \rightarrow U(1)$ such that $\psi_\xi^1(q) = q \cdot R_\xi(\Pi(q))$ for all $q \in P$.

Assume that the condition on Φ derived in the previous section has been satisfied over the fixed point z . Then $R_\xi(z) = 1$. It remains to show that R_ξ is constant over M . Let $u(s)$ be an arbitrary path in M , where $s \in [0, 1]$. We will show that R_ξ is constant over $u(s)$.

Recall that the determinant map $\text{Mp}^c(V) \xrightarrow{\eta} U(1)$ acts on $\lambda \in U(1) \subset \text{Mp}^c(V)$ by $\eta(\lambda) = \lambda^2$. Let $Y \xrightarrow{\pi} M$ be the circle bundle associated to $P \xrightarrow{\Pi} M$ by η , and let $P \xrightarrow{H} Y$ be the corresponding bundle map. It is easily verified that γ is basic with respect to H . Let γ^η be the $\mathfrak{u}(1)$ -valued one-form on Y such that $H^*\gamma^\eta = 2\gamma$. Then γ^η is a connection one-form on Y , and $d\gamma^\eta = \frac{2}{i\hbar}\pi^*\omega$.

The pushforward $H_*\xi_P$ is a well-defined vector field on Y , which we denote by ξ_Y . Let the flow of ξ_Y on Y be $\hat{\chi}_\xi^t$. The various vector fields and their flows are summarized below.

$$\begin{array}{ccccc}
 \xi_P, \psi_\xi^t & (P, \gamma) \xrightarrow{H} & (Y, \gamma) & & \xi_Y, \hat{\chi}_\xi^t \\
 & \downarrow & \downarrow & & \\
 \tilde{\xi}_M, \tilde{\phi}_\xi^t & \text{Sp}(M, \omega) & & & \\
 & \downarrow & \downarrow & & \\
 \xi_M, \phi_\xi^t & (M, \omega) \xrightarrow{=} & (M, \omega) & & \xi_M, \phi_\xi^t.
 \end{array}$$

Since $H_*\xi_P = \xi_Y$, H intertwines the flows ψ_ξ^t and $\hat{\chi}_\xi^t$. In particular,

$$\hat{\chi}_\xi^1 \circ H(q) = H \circ \psi_\xi^1(q) = H(q \cdot R_\xi(\Pi(q))) = H(q) \cdot R_\xi^2(\Pi(q)).$$

That is, the map $\hat{\chi}_\xi^1$ acts on Y by

$$\hat{\chi}_\xi^1(y) = y \cdot R_\xi^2(\pi(y)), \quad \forall y \in Y.$$

From the definitions of ξ_Y and γ^η , it follows that $\gamma^\eta(\xi_Y) = -\frac{2}{i\hbar}\pi^*\Phi^\xi$. Let ∂_θ be the vertical vector field on Y such that $\gamma^\eta(\partial_\theta) = 2\pi i \in \mathfrak{u}(1)$. Then we have

$$\xi_Y = (\xi_M)_{hor} + \frac{2}{h}\Phi^\xi\partial_\theta,$$

where $(\xi_M)_{hor}$ represents the lift of ξ_M to Y that is horizontal with respect to γ^η .

Let $C = \mathbb{R}/\mathbb{Z} \times [0, 1]$, with coordinates (r, s) . Define $F : C \rightarrow M$ by

$$F(r, s) = \exp(r\xi) \cdot u(s), \quad \forall (r, s) \in C,$$

and let $\omega_C = F^*\omega$. Construct the pullback of (Y, γ^η) to C :

$$\begin{array}{ccc} (D, \delta) & \xrightarrow{\widehat{F}} & (Y, \gamma^\eta) \\ \downarrow & & \downarrow \pi \\ (C, \omega_C) & \xrightarrow{F} & (M, \omega) \end{array}$$

where $\delta = \widehat{F}^*\gamma^\eta$ and

$$D = F^*Y = \{(c, y) \in C \times Y : F(c) = \pi(y)\}.$$

The bundle map $\widehat{F} : D \rightarrow Y$ acts by

$$\widehat{F}(c, y) = y, \quad \forall (c, y) \in D.$$

By construction, (D, δ) is a circle bundle with connection one-form over (C, ω_C) . Let ∂_θ also denote the vertical vector field on D such that $\delta(\partial_\theta) = 2\pi i \in \mathfrak{u}(1)$, and note that $\widehat{F}_*\partial_\theta = \partial_\theta$.

Abbreviate the vector fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial s}$ on C by ∂_r and ∂_s , with flows ψ_r^t and ψ_s^t respectively. It is immediate from the definition of F that

$$F_*|_c \partial_r = \xi_M(F(c)), \quad \forall c \in C.$$

Let $\Psi : C \rightarrow \mathbb{R}$ be given by

$$\Psi(r, s) = \Phi^\xi(u(s)), \quad \forall (r, s) \in C.$$

We claim that $\omega_C = \frac{\partial \Psi}{\partial s} dr \wedge ds$. This is established by calculating, at arbitrary $c = (r, s) \in C$,

$$\partial_r \lrcorner (\omega_C)_c = F^* d\Phi_{F(c)}^\xi,$$

and

$$\partial_s \lrcorner F^* d\Phi_{F(c)}^\xi = \left. \frac{\partial \Psi}{\partial s} \right|_{(r, s)}.$$

Let ζ_r be the vector field on D given by

$$\zeta_r = (\partial_r)_{hor} + \frac{2}{h} \Psi \partial_\theta,$$

with flow $\widehat{\psi}_r^t$. Then for all $(c = (r, s), y) \in D$,

$$\widehat{F}_*|_{(c, y)} \zeta_r = (\xi_M)_{hor}(\widehat{F}(c, y)) + \frac{2}{h} \Phi^\xi(u(s)) \partial_\theta = \xi_Y(\widehat{F}(c, y)).$$

Thus \widehat{F} intertwines the flows $\widehat{\psi}_r^t$ of ζ_r and $\widehat{\chi}_\xi^t$ of ξ_Y . In particular, at $t = 1$,

$$\widehat{F} \circ \widehat{\psi}_r^1(c, y) = \widehat{\chi}_\xi^1 \circ \widehat{F}(c, y) = \widehat{\chi}_\xi^1(y) = y \cdot R_\xi^2(\pi(y)) = \widehat{F}(c, y \cdot R_\xi^2(\pi(y))),$$

which implies that

$$\widehat{\psi}_r^1(c, y) = (c, y \cdot R_\xi^2(\pi(y))) = (c, y) \cdot R_\xi^2(\pi(y)).$$

That is, $\widehat{\psi}_r^1$ fixes the base C and rotates each fiber $D_{(c, y)}$ by $R_\xi^2(\pi(y))$. Note that $\pi(y) = F(c)$, so we have

$$R_\xi^2(\pi(y)) = R_\xi^2(F(c)) = R_\xi^2(\exp(r\xi)u(s)).$$

Let $\zeta_s = (\partial_s)_{hor}$ on D , with flow $\widehat{\psi}_s^t$. A standard calculation establishes that $[\zeta_r, \zeta_s] = 0$. Hence their flows $\widehat{\psi}_r^t$ and $\widehat{\psi}_s^t$ commute. In particular, for any $(c = (r, s), y) \in D$,

$$\widehat{\psi}_r^1 \circ \widehat{\psi}_s^t(c, y) = \widehat{\psi}_s^t \circ \widehat{\psi}_r^1(c, y) = \widehat{\psi}_s^t(c, y \cdot R_\xi^2(\pi(y))) = \widehat{\psi}_s^t(c, y) \cdot R_\xi^2(\exp(r\xi) \cdot u(s)),$$

having used the fact that $\widehat{\psi}_s^t$ is the flow of a horizontal vector field on D and is therefore equivariant with respect to the principal $U(1)$ action. We also calculate

$$\begin{aligned} \widehat{\psi}_r^1 \circ \widehat{\psi}_s^t(c, y) &= \widehat{\psi}_s^t(c, y) \cdot R_\xi^2(\pi(\widehat{F}(\widehat{\psi}_s^t(c, y)))) = \widehat{\psi}_s^t(c, y) \cdot R_\xi^2(F(\psi_s^t(c))) \\ &= \widehat{\psi}_s^t(c, y) \cdot R_\xi^2(\exp(r\xi) \cdot u(s+t)), \end{aligned}$$

where we note that $\psi_s^t(c) = (r, s+t)$. We conclude that $R_\xi^2(\exp(r\xi) \cdot u(s)) = R_\xi^2(\exp(r\xi) \cdot u(s+t))$ for all r, s, t . Hence R_ξ^2 is constant over the path u . Since u was arbitrary, R_ξ^2 is in fact constant on M .

Recall that $R_\xi(z) = 1$. Thus $R_\xi^2 = 1$ everywhere on M , and consequently $R_\xi = 1$ everywhere on M , as needed. Hence the T^k action is well defined everywhere on P . We have now proved the following theorem.

Theorem 3.1. Let (M, ω) have an effective Hamiltonian T^k action with momentum map Φ and a fixed point z . Then (M, ω, Φ) admits an equivariant metaplectic-c prequantization if and only if (M, ω) is metaplectic-c prequantizable and the momentum map Φ satisfies

$$\frac{1}{h}\Phi(z) - \frac{1}{2}w_{z^*} \in \mathbb{Z}^{k^*} \subset \mathfrak{t}^*.$$

Assume the hypotheses of the theorem, and let z be a fixed point for the T^k action on (M, ω) . By adding a constant to the momentum map Φ , it is always possible to satisfy the condition $\frac{1}{h}\Phi(z) - \frac{1}{2}w_{z^*} \in \mathbb{Z}^{k^*}$. Thus any metaplectic-c prequantization (P, Σ, γ) for (M, ω) can be converted to an equivariant metaplectic-c prequantization by a suitable shift of the momentum map.

4 Quantized Energy Levels

In this section, we extend the concept of a quantized energy level to the equivariant metaplectic-c prequantized system $(P, \Sigma, \gamma) \rightarrow (M, \omega, \Phi)$. Section 4.1 reviews the constructions due to Robinson [5] that we use to define a quantized energy level, and concludes with the generalization of our definition from [7] to a regular value of the momentum map Φ . In Section 4.2, we determine the quantized energy levels of the system (M, ω, Φ) , assuming that Φ has been shifted so that the metaplectic-c prequantization is equivariant.

4.1 Descending to the quotient

Assume that (M, ω) is metaplectic-c prequantizable, and let (P, Σ, γ) be a metaplectic-c prequantization:

$$(P, \gamma) \xrightarrow{\Sigma} \text{Sp}(M, \omega) \xrightarrow{\rho} (M, \omega), \quad \rho \circ \Sigma = \Pi, \quad d\gamma = \frac{1}{i\hbar}\Pi^*\omega.$$

Assume that there is an effective Hamiltonian T^k action on M with momentum map Φ and at least one fixed point. Shift Φ if necessary so that (P, Σ, γ) is an equivariant metaplectic-c prequantization for (M, ω, Φ) .

Let $x \in \mathfrak{t}^*$ be a regular value of the momentum map Φ , and let $S = \Phi^{-1}(x)$. Then S is a codimension- k embedded submanifold of M . Recall that $\{\xi_1, \dots, \xi_k\}$ is the standard basis for

\mathbb{R}^k . For all $s \in S$, the symplectic orthogonal to $T_s S$ is $T_s S^\perp = \text{span}\{\xi_{1M}(s), \dots, \xi_{kM}(s)\} \subset T_s S$, implying that S is a coisotropic submanifold of M . In the special case where $k = n$, S is a Lagrangian submanifold.

Within the model symplectic vector space (V, Ω) , let $W \subset V$ be a coisotropic subspace of codimension k , with symplectic orthogonal $W^\perp \subset W$. Then W/W^\perp is a symplectic vector space with a symplectic structure inherited from V . Let $\text{Sp}(V; W) \subset \text{Sp}(V)$ be the subgroup that preserves W . There is a natural group homomorphism $\text{Sp}(V; W) \xrightarrow{\nu} \text{Sp}(W/W^\perp)$.

Let $\text{Mp}^c(V; W) \subset \text{Mp}^c(V)$ be the preimage of $\text{Sp}(V; W)$ under σ , and let $\text{Mp}^c(W/W^\perp)$ be the metaplectic-c group for W/W^\perp . Robinson and Rawnsley [6] showed that ν lifts to a group homomorphism $\hat{\nu}$ on the level of metaplectic-c groups:

$$\begin{array}{ccccc} \text{Mp}^c(V) & \xleftarrow{\supset} & \text{Mp}^c(V; W) & \xrightarrow{\hat{\nu}} & \text{Mp}^c(W/W^\perp) \\ \downarrow \sigma & & \downarrow \sigma & & \downarrow \sigma \\ \text{Sp}(V) & \xleftarrow{\supset} & \text{Sp}(V; W) & \xrightarrow{\nu} & \text{Sp}(W/W^\perp) \end{array}$$

The lifted map $\hat{\nu}$ has the property that $\hat{\nu}_* \circ \eta_* = \eta_*$.

In the following construction, which is due to Robinson [5], the above diagram serves as a model for fiberwise constructions over (M, ω) . The first column corresponds to the original three-level structure,

$$(P, \gamma) \rightarrow \text{Sp}(M, \omega) \rightarrow (M, \omega).$$

For the second column, let $\text{Sp}(M, \omega; S) \subset \text{Sp}(M, \omega)$ be the subbundle, lying only over S , such that for all $s \in S$ and all $b \in \text{Sp}(M, \omega; S)_s$, $bW = T_s S$. Lastly, let (P^S, γ^S) be the pullback of (P, γ) to $\text{Sp}(M, \omega; S)$.

For the third column, treat W/W^\perp as a model symplectic vector space for the symplectic vector bundle $TS/TS^\perp \rightarrow S$, so that the symplectic frame bundle $\text{Sp}(TS/TS^\perp) \rightarrow S$ becomes a right principal $\text{Sp}(W/W^\perp)$ bundle over S . Then $\text{Sp}(TS/TS^\perp)$ is naturally identified with the bundle associated to $\text{Sp}(M, \omega; S)$ by the map ν . Let P_S be the bundle associated to P^S by the map $\hat{\nu}$. The properties of $\hat{\nu}$ guarantee that there is a one-form γ_S on P_S such that $\hat{\nu}^* \gamma_S = \gamma^S$. Then $(P_S, \gamma_S) \rightarrow \text{Sp}(TS/TS^\perp)$ is a principal circle bundle with connection one-form.

$$\begin{array}{ccccc} (P, \gamma) & \xleftarrow{\supset} & (P^S, \gamma^S) & \xrightarrow{\hat{\nu}} & (P_S, \gamma_S) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Sp}(M, \omega) & \xleftarrow{\supset} & \text{Sp}(M, \omega; S) & \xrightarrow{\nu} & \text{Sp}(TS/TS^\perp) \\ \downarrow & & \downarrow & & \downarrow \\ (M, \omega) & \xleftarrow{\supset} & S & \xrightarrow{=} & S \end{array}$$

Let $\xi \in \mathfrak{t}$ be arbitrary. Construct ξ_M with flow ϕ_ξ^t on M , $\tilde{\xi}_M$ with flow $\tilde{\phi}_\xi^t$ on $\text{Sp}(M, \omega)$, and ξ_P with flow ψ_ξ^t on P , as described in Section 3.1. Recall in particular that ξ_P is the lift of $\tilde{\xi}_M$ to P such that $\gamma(\xi_P) = -\frac{1}{i\hbar} \Pi^* \Phi^\xi$. Let $\hat{\xi}_M$ be the lift of $\tilde{\xi}_M$ to P that is horizontal with respect to γ , and let its flow be $\hat{\phi}_\xi^t$. Each of the vector fields ξ_M , $\tilde{\xi}_M$, $\hat{\xi}_M$ and ξ_P restricts to a vector field on the appropriate manifold in the second column, and descends to a vector field on the manifold in the third column. Moreover, since the T^k action commutes with all of the principal actions, the T^k action on each level preserves the manifold in the second column, and descends to a T^k action on the manifold in the third column.

Viewing (P, γ) as a circle bundle with connection one-form over $\text{Sp}(M, \omega)$, let ∂_θ be the vertical vector field on P such that $\gamma(\partial_\theta) = 2\pi i \in \mathfrak{u}(1)$. Then $\xi_P = \widehat{\xi}_M + \frac{1}{h}\Phi^\xi \partial_\theta$. We also denote by ∂_θ the restriction of this vector field to P^S , and the induced vertical vector field on P_S . In each column, we have $\gamma(\partial_\theta) = \gamma^S(\partial_\theta) = \gamma_S(\partial_\theta) = 2\pi i \in \mathfrak{u}(1)$. Note that for all $s \in S$, $\Phi^\xi(s) = x \cdot \xi$, since S is the x -level set of Φ . Thus, in columns 2 and 3,

$$\xi_P = \widehat{\xi}_M + \frac{1}{h}(x \cdot \xi) \partial_\theta.$$

4.2 Generalized quantized energy levels

In our previous paper [7], we considered the system (M, ω, H) , where $H \in C^\infty(M)$ is viewed as a Hamiltonian energy function. Let $(P, \Sigma, \gamma) \rightarrow (M, \omega)$ be a metaplectic-c prequantization. Let E be a regular value of H , and use the embedded coisotropic submanifold $S = H^{-1}(E) \subset M$ to construct the three columns of three-level structures as described in the previous section. Let ξ_H be the Hamiltonian vector field for H on M , and let its lift to $\text{Sp}(M, \omega)$ be $\widetilde{\xi}_H$. Then ξ_H and $\widetilde{\xi}_H$ restrict to column 2 and descend to column 3. We define E to be a **quantized energy level** for the system (M, ω, H) if γ_S has trivial holonomy over all closed orbits of $\widetilde{\xi}_H$ on $\text{Sp}(TS/TS^\perp)$.

This definition has a natural generalization to a regular value of a family of k Poisson-commuting functions. Recall that $\{\xi_1, \dots, \xi_k\}$ is the standard basis for $\mathfrak{t} = \mathbb{R}^k$. In our context, the family of Poisson-commuting functions consists of the k components of the momentum map, $\Phi^{\xi_1}, \dots, \Phi^{\xi_k}$. The generalized definition is as follows.

Definition 4.1. Let (P, Σ, γ) be an equivariant metaplectic-c prequantization for (M, ω, Φ) . Let x be a regular value of Φ , and let $S = \Phi^{-1}(x)$. Let F be the distribution on $\text{Sp}(TS/TS^\perp)$ spanned by the vector fields $\{\widetilde{\xi}_{1M}, \dots, \widetilde{\xi}_{kM}\}$. Then x is a **quantized energy level** for (M, ω, Φ) if γ_S has trivial holonomy over all of the leaves of F .

The connection one-form γ_S is flat over each leaf of the distribution spanned by $\{\widetilde{\xi}_{1M}, \dots, \widetilde{\xi}_{kM}\}$. For the regular value x to be a quantized energy level, it suffices to ensure that γ_S has trivial holonomy over the orbits of each ξ_{jM} . Given any initial point $b \in \text{Sp}(TS/TS^\perp)$, the integral curve $\widetilde{\phi}_{\xi_j}^t(b)$ satisfies $\widetilde{\phi}_{\xi_j}^1(b) = b$. We need to show that the integral curves $\widehat{\phi}_{\xi_j}^t(q)$ of the horizontal lift $\widehat{\xi}_{jM}$ on P_S satisfy $\widehat{\phi}_{\xi_j}^1(q) = q$, for all $q \in P_S$.

As previously noted, the vector fields ξ_{jP} and $\widehat{\xi}_{jM}$ on P_S are related by

$$\xi_{jP} = \widehat{\xi}_{jM} + \frac{1}{h}(x \cdot \xi_j) \partial_\theta.$$

Since these two vector fields differ by a constant multiple of ∂_θ everywhere on P_S , their flows are related by

$$\psi_{\xi_j}^t(q) = \widehat{\phi}_{\xi_j}^t(q) \cdot e^{2\pi i t(x \cdot \xi_j)/h}, \quad \forall q \in P.$$

We know that $\tau = \exp(\xi) \in T^k$ acts on P_S by the map ψ_ξ^1 for all $\xi \in \mathfrak{t}$, and this is a well-defined T^k action. In particular, $\exp(\xi_j) = I \in T^k$, so $\psi_{\xi_j}^1$ is the identity map on P_S . We have

$$\psi_{\xi_j}^1(q) = q = \widehat{\phi}_{\xi_j}^1(q) e^{2\pi i (x \cdot \xi_j)/h}.$$

Thus $\widehat{\phi}_{\xi_j}^1(q) = q$ if and only if $e^{2\pi i (x \cdot \xi_j)/h} = 1$, which occurs when $x \cdot \xi_j = N_j h$ for some $N_j \in \mathbb{Z}$. This condition is satisfied for all ξ_j exactly when $x \in h\mathbb{Z}^{k*} \subset \mathfrak{t}^*$. We conclude the following result.

Theorem 4.2. Let (M, ω) have an effective Hamiltonian T^k action with momentum map Φ and at least one fixed point. Assume that (M, ω) is metaplectic-c prequantizable, and shift Φ by a constant if necessary so that (M, ω, Φ) admits an equivariant metaplectic-c prequantization. Then the regular value $x \in \mathfrak{t}^*$ for Φ is a quantized energy level for the system if and only if

$$x \in h\mathbb{Z}^{k*} \subset \mathfrak{t}^*.$$

An immediate consequence of this theorem arises in the context of symplectic reduction. Assume the hypotheses of Theorem 4.2, and let $x \in \mathfrak{t}^*$ be a regular value of Φ . Further assume that T^k acts freely on the level set $S = \Phi^{-1}(x)$. Then, by Marsden-Weinstein reduction, the space of orbits $M_0 = S/T^k$ is a manifold, and it acquires a symplectic form ω_0 . Let $S \xrightarrow{\varrho} M_0$ be the quotient map from S to its orbit space. Then $\varrho^*\omega_0 = \omega_S$, where ω_S is the pullback of ω to S .

Let $s \in S$ be arbitrary and let $\varrho(s) = m_0$. On the level of tangent spaces, we have the short exact sequence

$$0 \rightarrow T_s S^\perp \rightarrow T_s S \xrightarrow{\varrho_*|_s} T_{m_0} M_0 \rightarrow 0,$$

implying that $\varrho_*|_s$ induces a linear symplectic isomorphism between $T_s S/T_s S^\perp$ and $T_{m_0} M_0$. For all $\xi \in \mathfrak{t}$, the pushforward $\phi_{\xi*}^t : TM \rightarrow TM$ preserves both TS and TS^\perp , and so descends to a map on TS/TS^\perp . If we let $\tau = \exp(\xi) \in T^k$ act on TS/TS^\perp by the pushforward $\phi_{\xi*}^1$, the result is a T^k action on TS/TS^\perp . Using these observations, it is straightforward to verify that the tangent bundle TM_0 is naturally identified with the quotient $(TS/TS^\perp)/T^k$. From this it follows that the symplectic frame bundle $\text{Sp}(M_0, \omega_0)$ is naturally identified with the quotient $\text{Sp}(TS/TS^\perp)/T^k$.

$$\begin{array}{ccc} (P_S, \gamma_S) & & \\ \downarrow & & \\ \text{Sp}(TS/TS^\perp) & \xrightarrow{\quad /T^k \quad} & \text{Sp}(M_0, \omega_0) \\ \downarrow & & \downarrow \\ S & \xrightarrow[\quad /T^k \quad]{\quad \varrho \quad} & (M_0, \omega_0) \end{array}$$

The following fact was stated by Robinson [5]. Let $(Y, \gamma) \rightarrow Z$ be a principal circle bundle with connection one-form over an arbitrary manifold Z . Let F be a fibrating foliation of Z with leaf space Z_0 and quotient map $Z \xrightarrow{\varrho} Z_0$. Let the curvature of γ be ϖ . If γ has trivial holonomy over the leaves of F , then (Y, γ) descends to a principal circle bundle with connection one-form $(Y_0, \gamma_0) \rightarrow Z_0$ such that the curvature ϖ_0 of γ_0 satisfies $\varrho^*\varpi_0 = \varpi$.

In our case, the base manifold is $\text{Sp}(TS/TS^\perp)$, and the fibrating foliation is $F = \text{span} \{ \tilde{\xi}_{1M}, \dots, \tilde{\xi}_{kM} \}$. By Definition 4.1, if $x \in \mathfrak{t}^*$ is a quantized energy level for (M, ω, Φ) , then γ_S has trivial holonomy over the leaves of F . This is exactly the condition required for $(P_S, \gamma_S) \rightarrow \text{Sp}(TS/TS^\perp)$ to descend to a circle bundle

$$(P_0, \gamma_0) \xrightarrow{\Sigma_0} \text{Sp}(M_0, \omega_0),$$

where $P_0 = P_S/T^k$ and γ_0 is a connection one-form on P_0 such that the curvature of γ_S is the pullback of the curvature of γ_0 . It is easily checked that $(P_0, \Sigma_0, \gamma_0)$ is a metaplectic-c prequantization for (M_0, ω_0) . In other words, when x is a quantized energy level for (M, ω, Φ) , the top row of the diagram above can be completed in the obvious manner, and the result is a metaplectic-c prequantization for the symplectic reduction.

By Theorem 4.2, the quantized energy levels of (M, ω, Φ) are the regular values of Φ that lie in $h\mathbb{Z}^{k*}$. We conclude the following.

Theorem 4.3. Let (M, ω) have an effective Hamiltonian T^k action with momentum map Φ and at least one fixed point. Assume that (M, ω) is metaplectic-c prequantizable, and let (P, Σ, γ) be an equivariant metaplectic-c prequantization for (M, ω, Φ) . If $x \in \mathfrak{t}^*$ is a regular value of Φ that lies in $h\mathbb{Z}^{k*}$, and if T^k acts freely on the level set $S = \Phi^{-1}(x)$, then the symplectic reduction (M_0, ω_0) of this level set acquires a metaplectic-c prequantization by taking the quotient of (P_S, γ_S) by T^k .

5 Examples

5.1 Harmonic oscillators

Let $M = \mathbb{R}^{2n} = \mathbb{C}^n$, with Cartesian coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ and complex coordinates $z_j = q_j + ip_j$, $1 \leq j \leq n$. Equip M with the symplectic form $\omega = \sum_{j=1}^n dq_j \wedge dp_j = \frac{1}{2i} \sum_{j=1}^n d\bar{z}_j \wedge dz_j$. Since M is contractible, (M, ω) admits a metaplectic-c prequantization that is unique up to isomorphism.

Let the circle $T^1 = U(1)$ act on M as follows: given $\tau \in T^1$ and $m = (z_1, \dots, z_n) \in M$,

$$\tau \cdot m = (\tau z_1, \dots, \tau z_n).$$

Identify \mathfrak{t} with \mathbb{R} such that for all $\xi \in \mathfrak{t}$, $\exp(\xi) = e^{2\pi i \xi} \in T^1$. The momentum map $\Phi : M \rightarrow \mathfrak{t}^*$ is given, up to an additive constant, by

$$\Phi(m) = -\pi \sum_{j=1}^n |z_j|^2, \quad \forall m \in M.$$

The fixed point of the T^1 action is the origin, $z = (0, \dots, 0)$. If we identify $T_z M$ with $\mathbb{R}^{2n} = \mathbb{C}^n$ in the obvious way, it is immediate that every $\tau \in T^1$ acts on $T_z M$ as a complex linear isomorphism. Explicitly, for any $\xi \in \mathfrak{t}$, if $\tau = \exp(\xi)$, then τ acts on $T_z M$ by the complex matrix $\text{diag}(e^{2\pi i \xi}, \dots, e^{2\pi i \xi})$. The group homomorphism $w_z : T^k \rightarrow U(1)$, as defined in Section 3.2, is

$$w_z(\tau) = \text{Det}_{\mathbb{C}} \text{diag}(e^{2\pi i \xi}, \dots, e^{2\pi i \xi}) = e^{2\pi i n \xi}.$$

Therefore

$$\frac{1}{2} w_{z*} = \frac{n}{2} \in \mathfrak{t}^*.$$

Over the fixed point z , we find that

$$\frac{1}{h} \Phi(z) - \frac{1}{2} w_{z*} = -\frac{1}{2} w_{z*} = -\frac{n}{2}.$$

If n is even, then $-\frac{n}{2} \in \mathbb{Z}^* \subset \mathfrak{t}^*$ and the equivariance condition is satisfied, but not if n is odd. Let

$$\Phi' = \Phi + \frac{hn}{2}.$$

Then Φ' is also a momentum map for the T^1 action, and $\frac{1}{h} \Phi'(z) - \frac{1}{2} w_{z*} \in \mathbb{Z}^* \subset \mathfrak{t}^*$, for all n .

The quantized energy levels of the system (M, ω, Φ') are the regular values of Φ' that are in $h\mathbb{Z}^* \subset \mathfrak{t}^*$. Since

$$\Phi'(m) = -\pi \sum_{j=1}^n |z_j|^2 + \frac{hn}{2},$$

a regular value is $x \in \mathbb{R}$ such that $x < \frac{\hbar n}{2}$. Thus a quantized energy level takes the form

$$x = -\hbar N,$$

where $N \in \mathbb{Z}$ is such that $N > -\frac{n}{2}$.

The Hamiltonian energy function for an n -dimensional harmonic oscillator is

$$H = \frac{1}{2} \sum_{j=1}^n (q_j^2 + p_j^2) = \frac{1}{2} \sum_{j=1}^n |z_j|^2.$$

Note that $H = -\frac{1}{2\pi} \Phi' + \frac{\hbar n}{2}$. Therefore the quantized energy levels for the system (M, ω, H) take the form

$$E = \hbar \left(N + \frac{n}{2} \right),$$

where $N \in \mathbb{Z}$ is such that $N > -\frac{n}{2}$.

For comparison, the standard quantum mechanical calculation for the energy levels of the quantized harmonic oscillator yields

$$E = \hbar \left(N + \frac{n}{2} \right),$$

where $N \in \mathbb{Z}$ is such that $N \geq 0$. The two calculations do not agree on the starting point for the energy spectrum (but see below). However, the equivariant metaplectic-c prequantization does yield the $\frac{n}{2}$ shift in the energy levels. By contrast, Kostant-Souriau quantization requires the half-form correction to obtain this shift, which uses a choice of polarization.

In quantum mechanics, the quantized energy levels of this system are obtained by solving the Schrödinger equation, which is linear: an n -dimensional harmonic oscillator is equivalent to n independent one-dimensional harmonic oscillators. Consider the system described by the functions H_1, \dots, H_n where $H_j = \frac{1}{2}(q_j^2 + p_j^2)$ for each j . By an essentially identical calculation, we find that the quantized energy levels for such a system have the form (E_1, \dots, E_n) , where for each j , $E_j = \hbar(N_j + \frac{1}{2})$ for some $N_j \in \mathbb{Z}$ such that $N_j > -\frac{1}{2}$. If we view the quantized energy levels of the n -dimensional harmonic oscillator as $E = E_1 + \dots + E_n$, we obtain

$$E = \hbar \left(N + \frac{n}{2} \right), \quad N \in \mathbb{Z}, \quad N \geq 0.$$

We now have both the $\frac{n}{2}$ shift and the correct starting point, suggesting that this is the more appropriate mathematical interpretation of the physical system.

5.2 Complex projective space

Consider \mathbb{C}^{n+1} with the usual complex coordinates $z = (z_0, \dots, z_n)$, and complex projective space \mathbb{CP}^n with the usual homogeneous coordinates $[z] = [z_0 : \dots : z_n]$. The two-form

$$\varpi_{FS} = \partial \bar{\partial} \log(|z|^2)$$

on \mathbb{C}^{n+1} descends to \mathbb{CP}^n . Let

$$\omega_{FS} = K i \varpi_{FS}$$

on \mathbb{CP}^n , where $K > 0$ is a positive constant to be determined. Then ω_{FS} is a Kähler form on \mathbb{CP}^n : specifically, a scalar multiple of the Fubini-Study form.

Robinson and Rawnsley [6] demonstrated that $(\mathbb{CP}^n, \omega_{FS})$ admits a metaplectic-c prequantization if and only if $K = \hbar(N + \frac{n+1}{2})$ for some $N \in \mathbb{Z}$. Note that \mathbb{CP}^n admits metaplectic-c

prequantizations for all n . This is an improvement over the Kostant-Souriau quantization scheme with half-form correction, because \mathbb{CP}^n does not admit a metaplectic structure when n is even.¹

As a concrete example, we consider \mathbb{CP}^2 , which does not admit a metaplectic structure. For any K of the form $K = \hbar \left(N + \frac{3}{2}\right)$, $N \in \mathbb{Z}$, the symplectic manifold $(\mathbb{CP}^2, \omega_{FS})$ does not admit a prequantization line bundle either, since ω_{FS} is not integral. However, it does admit a metaplectic-c prequantization for any such K . We choose $K = \frac{3}{2}\hbar$.

Define an action of T^2 on \mathbb{C}^3 such that if $\tau = (\tau_1, \tau_2) \in T^2$, then

$$\tau \cdot z = (z_0, \tau_1^{k_1^1} \tau_2^{k_2^1} z_1, \tau_1^{k_1^2} \tau_2^{k_2^2} z_2),$$

where $k^j = (k_1^j, k_2^j) \in \mathbb{Z}^2$ for $j = 1, 2$, and $\{k^1, k^2\}$ is an integer basis for \mathbb{Z}^2 . This action descends to an effective Hamiltonian action of T^2 on $(\mathbb{CP}^2, \omega_{FS})$. A calculation establishes that the momentum map takes the form

$$\Phi([z]) = -\frac{3\hbar}{2} \left(k_1^1 \frac{|z_1|^2}{|z|^2} + k_1^2 \frac{|z_2|^2}{|z|^2}, k_2^1 \frac{|z_1|^2}{|z|^2} + k_2^2 \frac{|z_2|^2}{|z|^2} \right) + \hbar(C_1, C_2), \quad \forall [z] \in \mathbb{CP}^2,$$

where $(C_1, C_2) \in \mathbb{R}^{2*}$ is a constant.

The fixed points of the action are $Z_0 = [1, 0, 0]$, $Z_1 = [0, 1, 0]$ and $Z_2 = [0, 0, 1]$. It suffices to find a value of (C_1, C_2) such that the value of Φ at one of these points satisfies the equivariance condition in Theorem 3.1. For example, at Z_0 , we have

$$\frac{1}{\hbar} \Phi(Z_0) = (C_1, C_2).$$

The equivariance condition at this point is

$$\frac{1}{\hbar} \Phi(Z_0) - \frac{1}{2} w_{Z_0*} = (C_1, C_2) - \frac{1}{2} w_{Z_0*} \in \mathbb{Z}^{2*},$$

where $w_{Z_0} : T^k \rightarrow U(1)$ is defined in terms of the action of T^2 on the tangent space $T_{Z_0} \mathbb{CP}^2$. We can satisfy the equivariance condition by taking $(C_1, C_2) = \frac{1}{2} w_{Z_0*}$. It remains to calculate $\frac{1}{2} w_{Z_0*}$.

On the open set $\{[z] \in \mathbb{CP}^2 : z_0 \neq 0\}$, use local coordinates $(\zeta_1, \zeta_2) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)$ for \mathbb{CP}^2 . Let $\xi = \sum_{j=1}^2 a_j \xi_j \in \mathfrak{t}$ be arbitrary, and let $\tau = \exp(\xi) = (e^{2\pi i a_1}, e^{2\pi i a_2})$. Then τ acts on the point (ζ_1, ζ_2) by

$$\tau \cdot (\zeta_1, \zeta_2) = (e^{2\pi i(a_1 k_1^1 + a_2 k_2^1)} \zeta_1, e^{2\pi i(a_1 k_1^2 + a_2 k_2^2)} \zeta_2).$$

Identify $T_{Z_0} \mathbb{CP}^2$ with \mathbb{C}^2 in the natural way. Then the complex matrix corresponding to the action of τ on $T_{Z_0} \mathbb{CP}^2$ is

$$\text{diag} \left(\exp [2\pi i(a_1 k_1^1 + a_2 k_2^1)], \exp [2\pi i(a_1 k_1^2 + a_2 k_2^2)] \right).$$

Therefore

$$w_{Z_0}(\tau) = \exp [2\pi i(k_1^1 + k_1^2)a_1 + 2\pi i(k_2^1 + k_2^2)a_2],$$

which implies that

$$\frac{1}{2} w_{Z_0*} = \frac{1}{2} (k_1^1 + k_1^2, k_2^1 + k_2^2) \in \mathbb{R}^{2*}.$$

¹Some additional detail: if we consider a regular level set of the n -dimensional harmonic oscillator from Section 5.1 corresponding to energy K , the symplectic reduction at this level is the symplectic manifold $(\mathbb{CP}^{n-1}, Ki\varpi_{FS})$. The fact that the reduced system admits a metaplectic-c prequantization when $K = \hbar \left(N + \frac{n}{2}\right)$ can be checked directly using properties of \mathbb{CP}^{n-1} , as in [6], or it can be seen immediately by applying Theorem 4.3 to the n -dimensional harmonic oscillator.

Hence we set $(C_1, C_2) = \frac{1}{2}(k_1^1 + k_1^2, k_2^1 + k_2^2)$.

We can verify that this choice of (C_1, C_2) also satisfies the equivariance condition over Z_1 . At Z_1 , we have

$$\frac{1}{h}\Phi(Z_1) = -\frac{3}{2}(k_1^1, k_2^1) + \frac{1}{2}(k_1^1 + k_1^2, k_2^1 + k_2^2).$$

We need to calculate $\frac{1}{2}w_{Z_1*}$, where w_{Z_1} is defined in terms of the T^2 action on the tangent space $T_{Z_1}\mathbb{CP}^2$. On the open set $\{[z] \in \mathbb{CP}^2 : z_1 \neq 0\}$, use local coordinates $(\zeta_0, \zeta_2) = \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right)$. Then $\tau = (e^{2\pi i a_1}, e^{2\pi i a_2})$ acts by

$$\tau \cdot (\zeta_0, \zeta_2) = (e^{2\pi i(-a_1 k_1^1 - a_2 k_2^1)} \zeta_0, e^{2\pi i(a_1(k_1^2 - k_1^1) + a_2(k_2^2 - k_2^1))} \zeta_2).$$

An identical calculation to the one performed at Z_0 yields

$$\frac{1}{2}w_{Z_1*} = \frac{1}{2}(k_1^2 - 2k_1^1, k_2^2 - 2k_2^1).$$

Now,

$$\begin{aligned} \frac{1}{h}\Phi(Z_1) - \frac{1}{2}w_{Z_1*} &= -\frac{3}{2}(k_1^1, k_2^1) + \frac{1}{2}(k_1^1 + k_1^2, k_2^1 + k_2^2) - \frac{1}{2}(k_1^2 - 2k_1^1, k_2^2 - 2k_2^1) \\ &= (0, 0) \in \mathbb{Z}^{2*}, \end{aligned}$$

as needed. One can similarly check Z_2 .

The image of the momentum map in \mathbb{R}^{2*} (scaled by $\frac{1}{h}$, for simplicity) is the triangle with vertices

$$\begin{aligned} \frac{1}{h}\Phi(Z_0) &= \frac{1}{2}(k_1^1 + k_1^2, k_2^1 + k_2^2), & \frac{1}{h}\Phi(Z_1) &= \frac{1}{2}(k_1^2 - 2k_1^1, k_2^2 - 2k_2^1), \\ \frac{1}{h}\Phi(Z_2) &= \frac{1}{2}(k_1^1 - 2k_1^2, k_2^1 - 2k_2^2). \end{aligned}$$

The quantized energy levels correspond to the integer lattice points lying strictly in the interior of the triangle.

As a particularly simple example, consider $k^1 = (1, 0)$ and $k^2 = (0, 1)$. The vertices of the image of the momentum map are

$$\frac{1}{h}\Phi(Z_0) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \frac{1}{h}\Phi(Z_1) = \left(-1, \frac{1}{2}\right), \quad \frac{1}{h}\Phi(Z_2) = \left(\frac{1}{2}, -1\right).$$

There is exactly one integer lattice point in the interior of the triangle, namely $(0, 0)$. More generally, if we let $K = \hbar(N + \frac{3}{2})$ for an arbitrary $N \in \mathbb{Z}$, $N \geq 0$, the vertices are

$$\frac{1}{h}\Phi(Z_0) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \frac{1}{h}\Phi(Z_1) = \left(-N-1, \frac{1}{2}\right), \quad \frac{1}{h}\Phi(Z_2) = \left(\frac{1}{2}, -N-1\right),$$

and the number of quantized energy levels is $\frac{(N+2)(N+1)}{2} = \binom{N+2}{N}$. If $N = -1$, the system is metaplectic-c prequantizable and the vertices take the form given above, but there are no quantized energy levels.

The symplectic reduction of the three-dimensional harmonic oscillator at the quantized energy level $K = \hbar(N + \frac{3}{2})$ is exactly the symplectic manifold $(\mathbb{CP}^2, Ki\varpi_{FS})$. We recognize the value $\binom{N+2}{N}$ from the quantum mechanical calculation as the multiplicity of the N th quantized energy level for $N \geq 0$. This calculation does not yield a quantized energy level corresponding to $N = -1$, and indeed this regular value has multiplicity zero by the above interpretation.

This last example and that in Section 5.1 are different facets of the same system. We will treat the relationships between them in greater generality in a subsequent paper concerning equivariant metaplectic-c prequantizations and quantized energy levels for toric manifolds (in preparation).

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